Tilings and Geometric problems
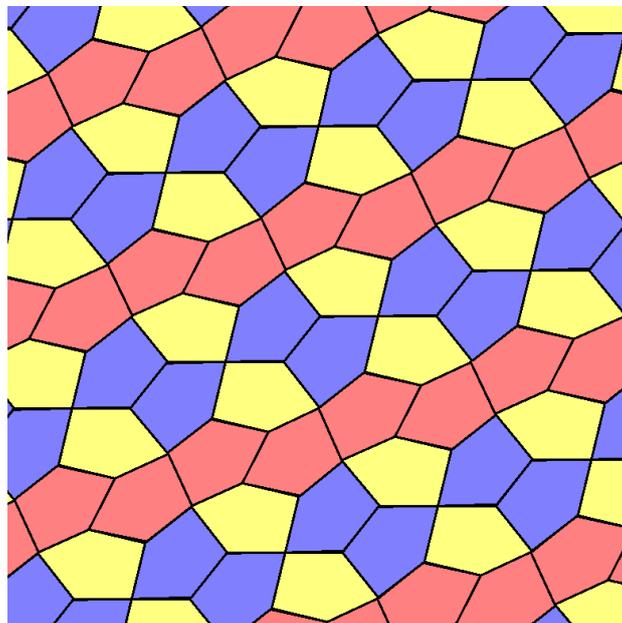
G4G11 gift article by Jaap Scherphuis

At the previous Gathering for Gardner I presented many beautiful tiling patterns. These were periodic tiling patterns that each use just one polygonal tile shape, which I found during my search for new tilable convex pentagon shapes. You can find the tilings at http://www.jaapsch.net/tilings/

Here I will show a few neat tilings and some geometry problems that arise from them.

1. A tiling by Marjorie Rice

Marjorie Rice created a fascinating tiling pattern, which to my surprise was not found by my search algorithms.

It looks like the single tile shape shown on the right can be used to tile the plane, without even needing to flip any of the tiles over.

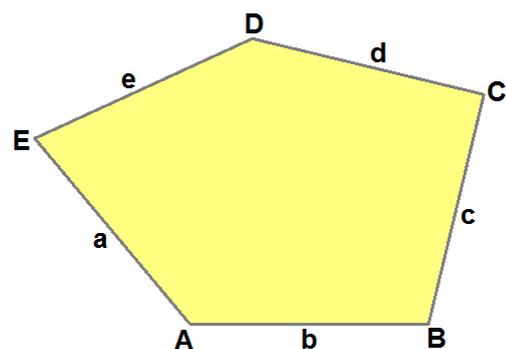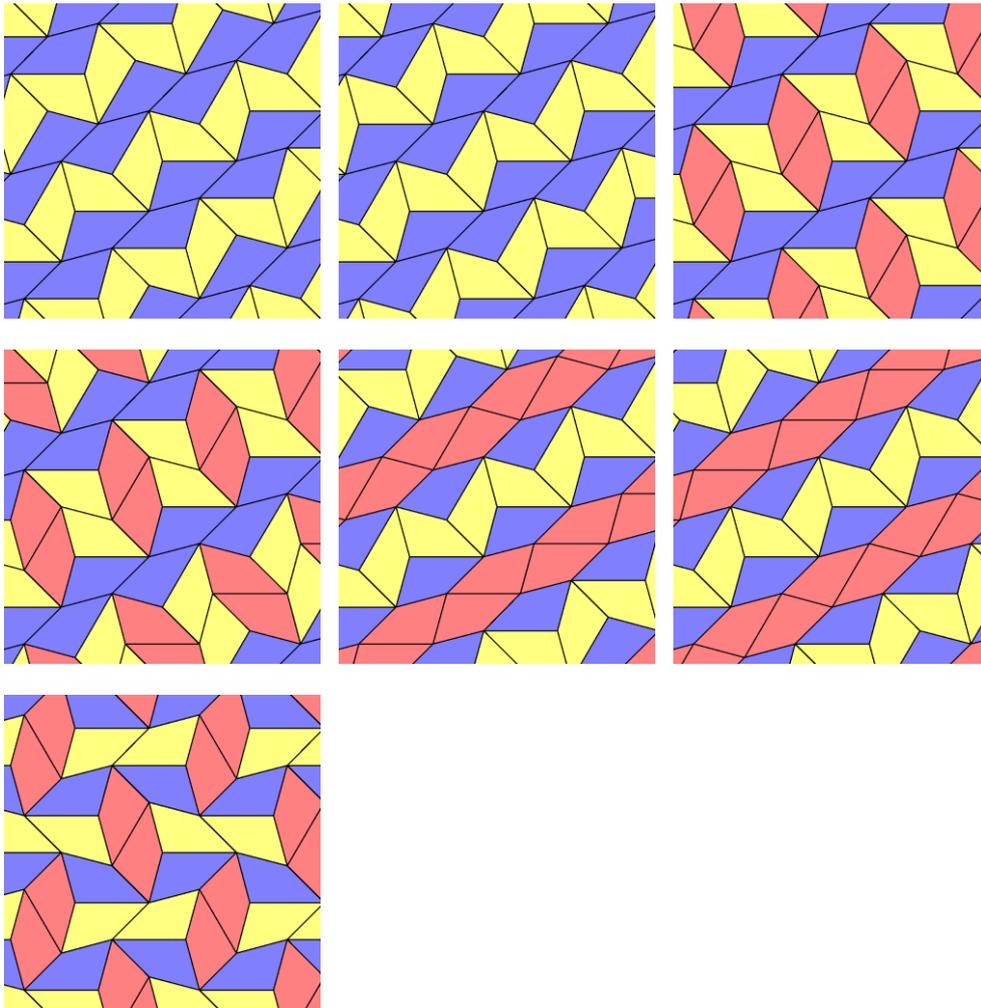Prove that this tiling is not actually possible (at least, not with identical tiles).

Figure 1: The impossible pentagonal tile

2. A quadrilateral

Here is family of very nice tilings, all involving a single type of quadrilateral.

The three sides a, c, and d are the same length, and the angles A and C satisfy the equation 2A+C=360.
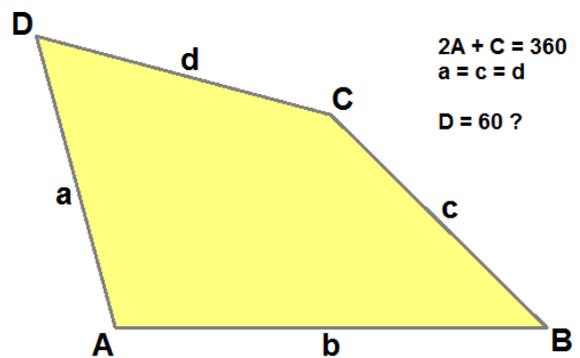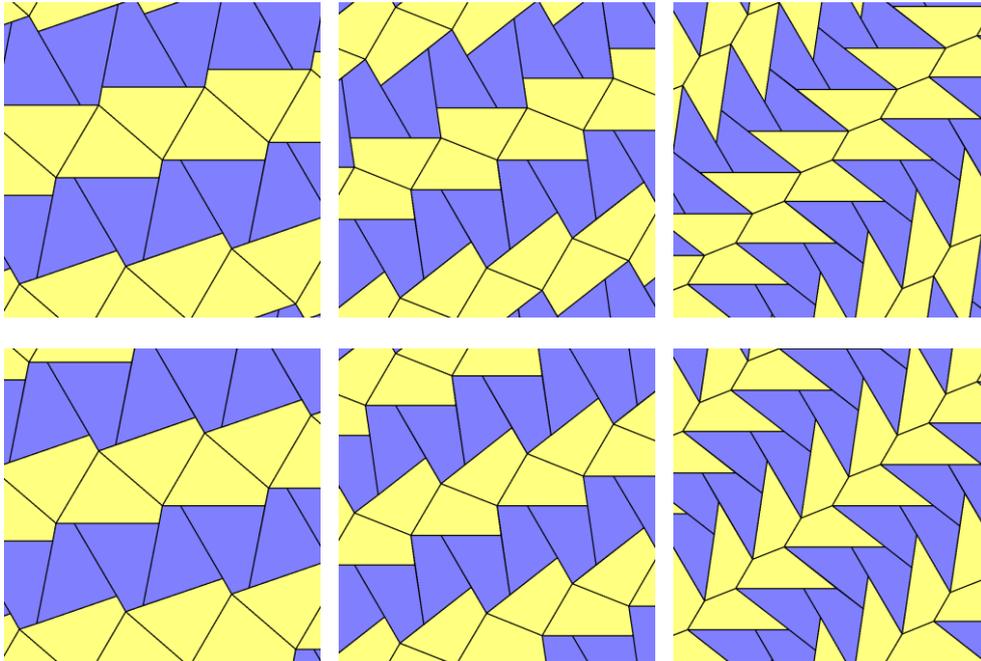
Prove that angle D is 60 degrees.

Figure 2: The first quadrilateral tile

3. Another quadrilateral

Here is another type of quadrilateral. This one admits the two 2-isohedral tilings, shown here three times using different choices for the single degree of freedom this tile has.

This quadrilateral satisfies the relations a=d, b=c+d, B+D=180.
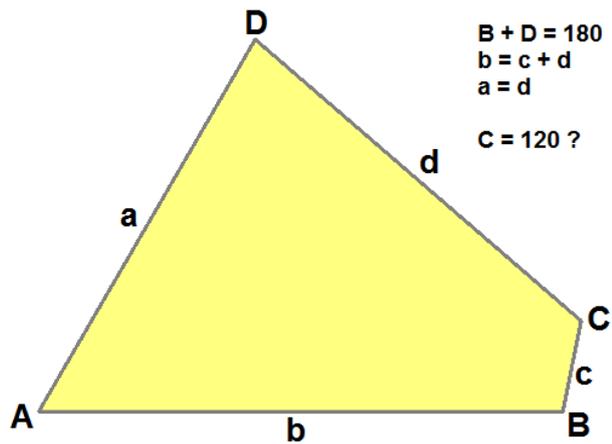
Prove that angle C is 120 degrees.

B + D = 180
b = c + d
a = d

C = 120 ?

Figure 3: The second quadrilateral tile

Answers:

1. By examining which edges touch, it is easy to see that all the edges of the tile must be the same length. By looking which tile corners meet at the vertices in the tiling, we find that the angles must satisfy:

    1) 2A+B=360
    2) B+E=180
    3) B+2C+E=360
    4) A+C+D=360
    5) 2D+E=360

    From 2 and 3 we get that C is 90 degrees.
    We know that a=e, i.e. ADE is an isosceles triangle.
    Therefore the angles ADE and DAE are equal, and are equal to (180-E)/2.
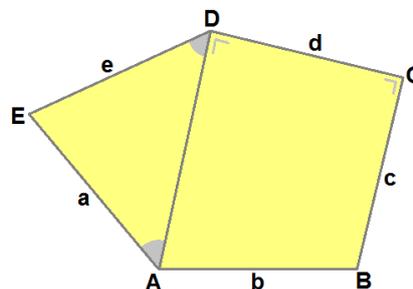   CDA = D-ADE = D-(180-E)/2 = (2D-180+E)/2
    Substituting (5) we get CDA = (2D-180+E)/2 = CDA = (360-180)/2 = 90.
    Considering the quadrilateral ABCD, it has two right angles (C and CDA), so two opposite sides are parallel and the distance between them is the edge length d. The fourth side b is known to be the same length as d, so the only way for it to span the distance between the parallel sides if it was also perpendicular, i.e. if ABCD were a square.
    Now that B=90, the equations easily show that A=D=135, B=C=E=90.
    This set of angles then clearly forces sides a and e to be shorter than b, c, and d (by a factor of 1/√2), whereas they should all be equal.
    This contradiction shows that the tiling is not possible with tiles that are of exactly the same shape. A slight fudging of the edge lengths allows B to have the obtuse angle needed for the tiling to seem to work.

2. Consider the circle with centre C and radius c=d. The points B and D obviously lie on this circle. The arc BD of the circle is subtended by an angle of 360-C. We know that A = (360-C)/2, so angle BAD is exactly half the central angle of the arc BD. This can only happen if A lies on the circle. Therefore AC=CD=AD and ACD is an equilateral triangle, and D must be 60 degrees.
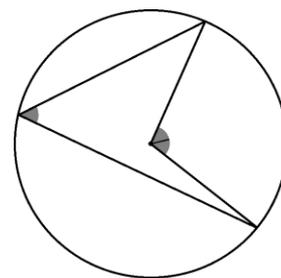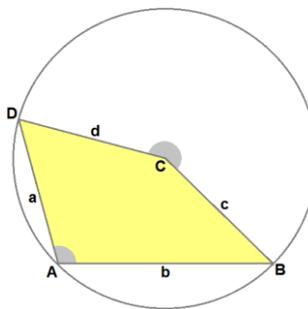
    Figure 4: Subtended angle is half the central angle

3. B+D=180, so the quadrangle is cyclic. Draw the circle that goes through the points A,B,C,D, and draw diagonal BD.
    Since CD=AD, the corresponding arcs on the circle are equal.
    The angles CBD and DBA subtend equal arcs so are equal too.
    Draw point X on AB such that BC=BX.
    Triangles DBC and DBX are congruent (they share 2 sides and an angle), so CD=DX.
    We're given that AB=BC+AD, so AX = AB-BX = BC+AD-BX = AD = CD = DX
    Therefore triangle DAX is equilateral and angle A=60.
    C = 360-A-B-D = 360-180-60 = 120.