Pentagon Tilings

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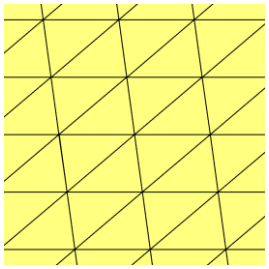
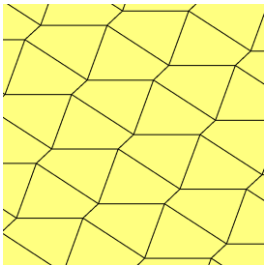
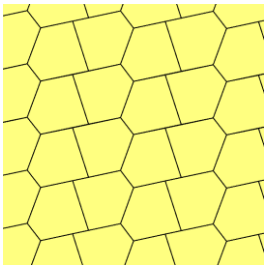
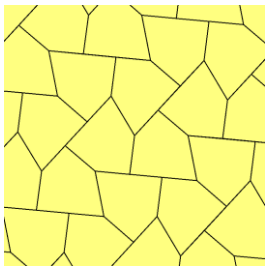
It is well known that any triangle can tile the plane, as can any quadrangle, convex or not. Much less is known about pentagons. A regular pentagon does not tile the plane, but various non-regular pentagons do.

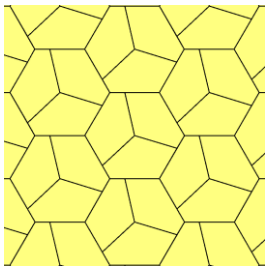

Any triangle tiles the plane

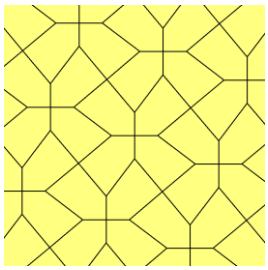

Any quadrangle tiles the plane

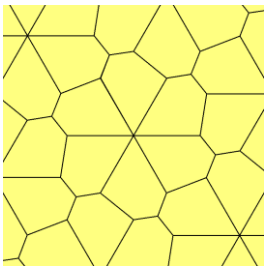

Type 1 pentagon tiles the plane
(two adjacent angles add to 180)

In 1975 Martin Gardner wrote an article about which types of convex pentagons can tile the plane. Richard B. Kershner had attempted to enumerate them, and thought that his list of 8 types was complete until in a reaction to Gardner's article Richard E. James III wrote in with another tilable convex pentagon. Soon after, Marjorie Rice found four others, bringing the total to 13. Only one further type has been found since, by Rolf Stein in 1985. It is still not known whether this list of fourteen types is now complete. (1)

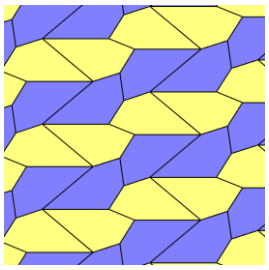

Type 2


Type 3

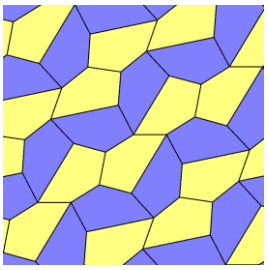

Type 4


Type 5

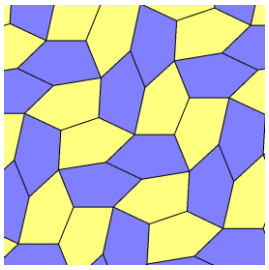
The first 5 types of convex pentagon form isohedral tilings, which means that by using the symmetries of the tiling as a whole, any tile can be mapped onto any other tile. Isohedral tilings are fully understood and the 24 isohedral tilings with convex pentagons all use pentagons of these five types. (2)



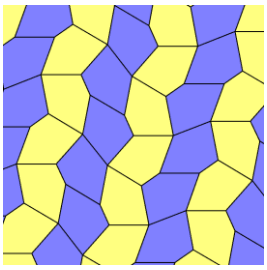
Type 6 (Kershner)



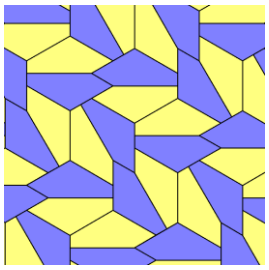
Type 7 (Kershner)



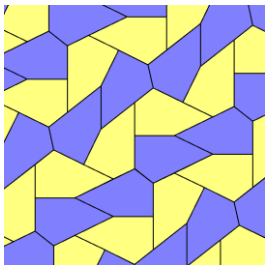
Type 8 (Kershner)



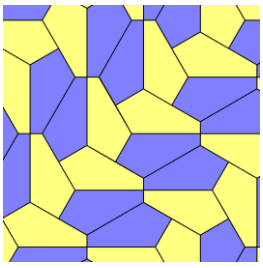
Type 9 (Rice)



Type 11 (Rice)

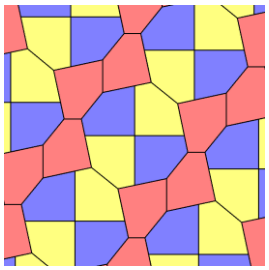


Type 12 (Rice)

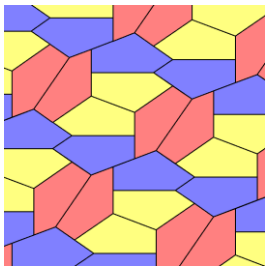


Type 13 (Rice)

The seven other tiles found by Kershner and Rice are 2-isohedral, while the remaining two are 3-isohedral. In the figures the isohedrality classes are shown with different colours, and it is easy to verify that any two tiles of the same colour can be mapped to one another by some symmetry of the tiling, whereas two tiles of different colours cannot.

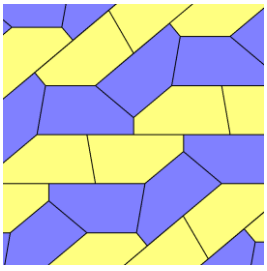


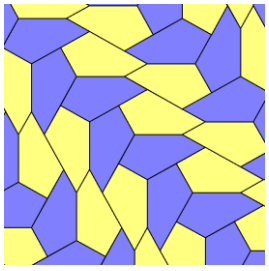
Type 10 (James)

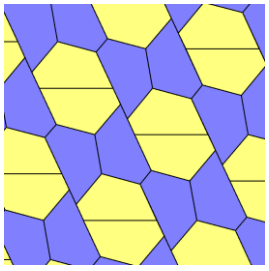


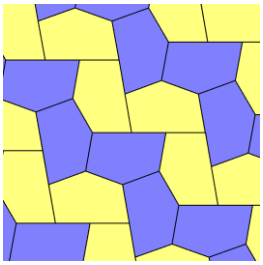
Type 14 (Stein)

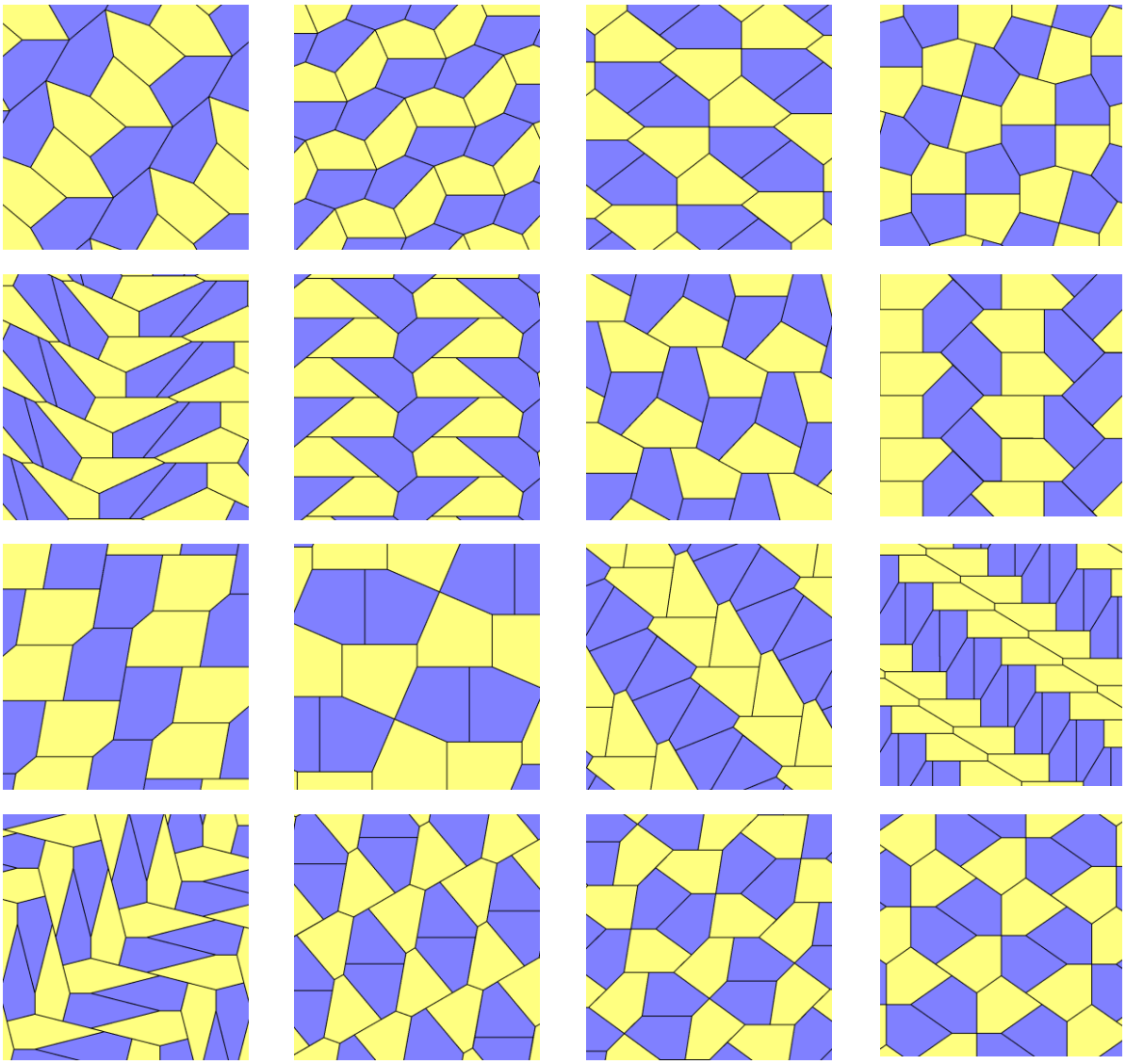
I have generated by computer all possible 2-isohedral pentagon tilings where the two types of tiles occur in a 1:1 ratio. This has resulted in many interesting tilings, some of which are illustrated here. Most of the tilings used tiles that were special cases of type 1, i.e. they have two parallel sides.



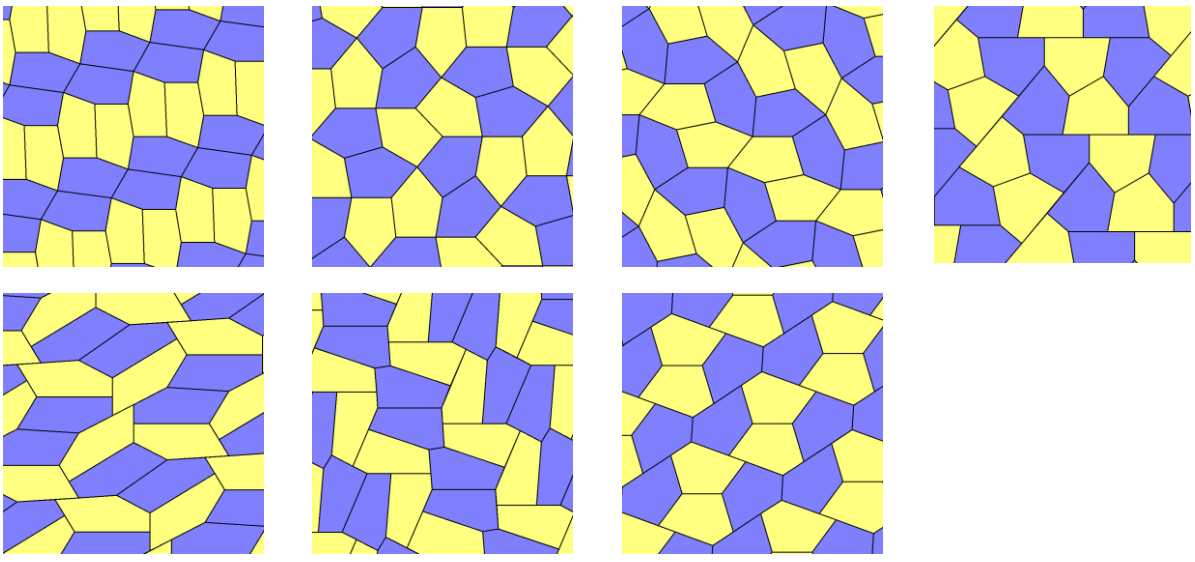




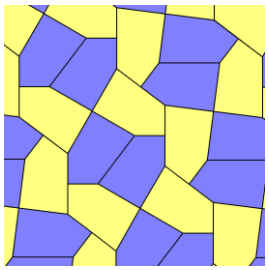




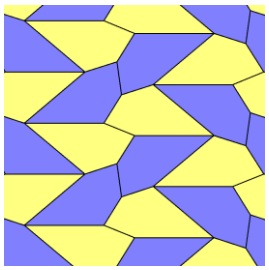
Seven kinds of tiling were found that have tiles that are special cases of tile type 2:



A few of the tilings above can be found on Marjorie Rice's website (3), but I have not found any other sources for the rest of them. Lastly, I found an alternative tiling for the type 6 tile, as well as one tiling with a special case of a type 4 tile.

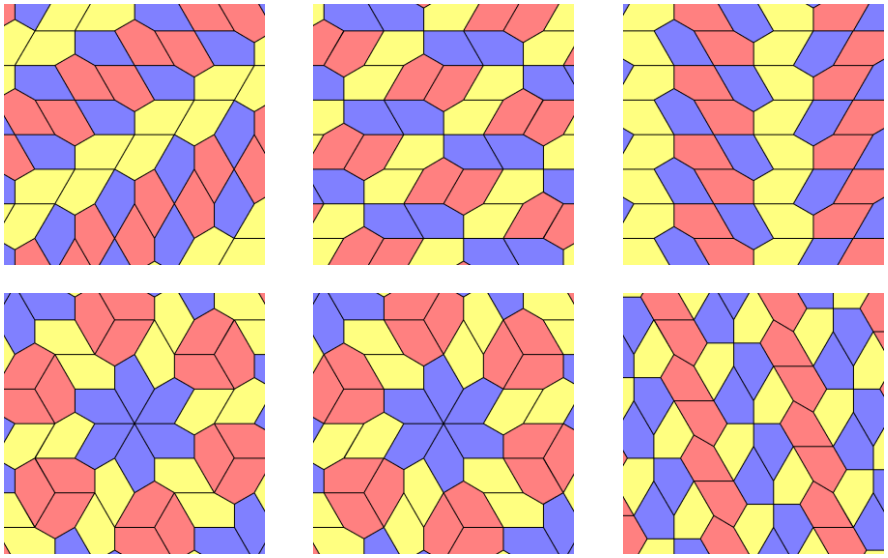


Type 4 special case



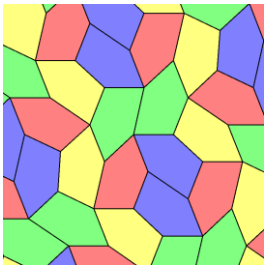
Type 6 variation

Assuming that there were no mistakes or bugs in the programs, this means that any further tile types must be found in k-isohedral tilings with k at least 3. I intend eventually to exhaustively search all 3-isohedral tilings, but at the moment I have only searched for those that are edge-to-edge.



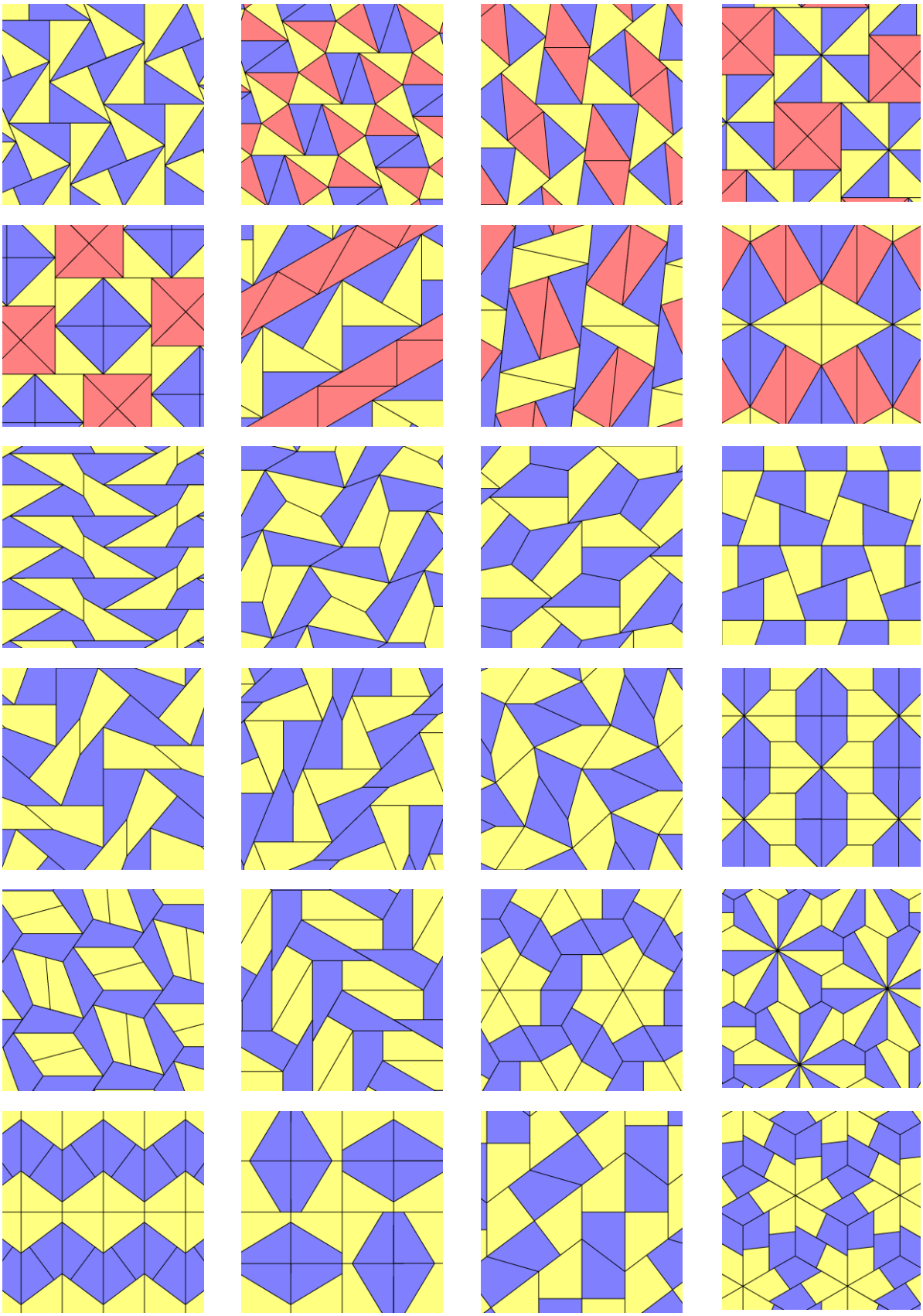
Rice (3)

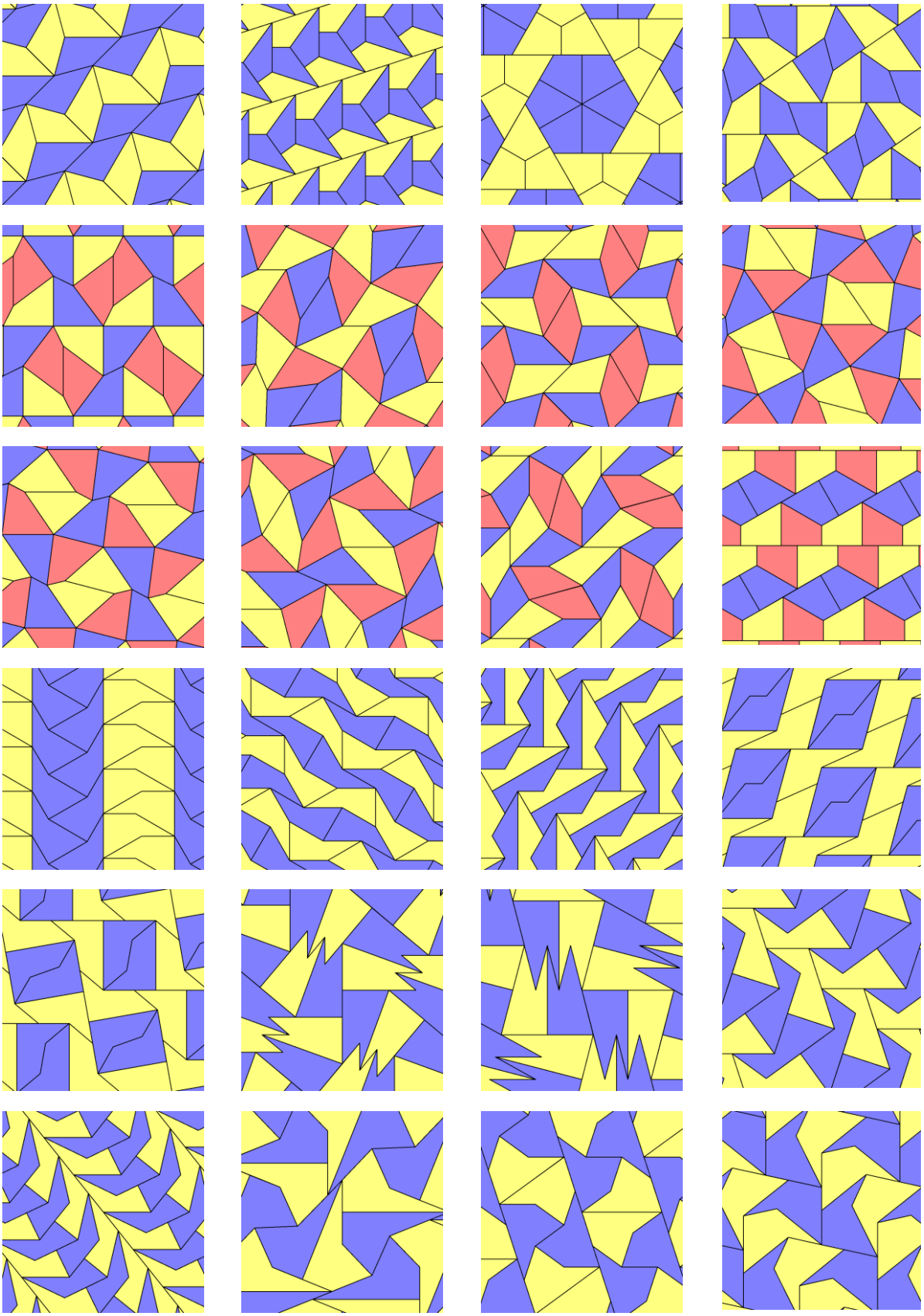
The space of all 4-isohedral tilings is too large to exhaustively search, even if restricted to edge-to-edge tilings, but a partial search did encounter the following amazing tiling, which uses a special case of the type 8 tile.

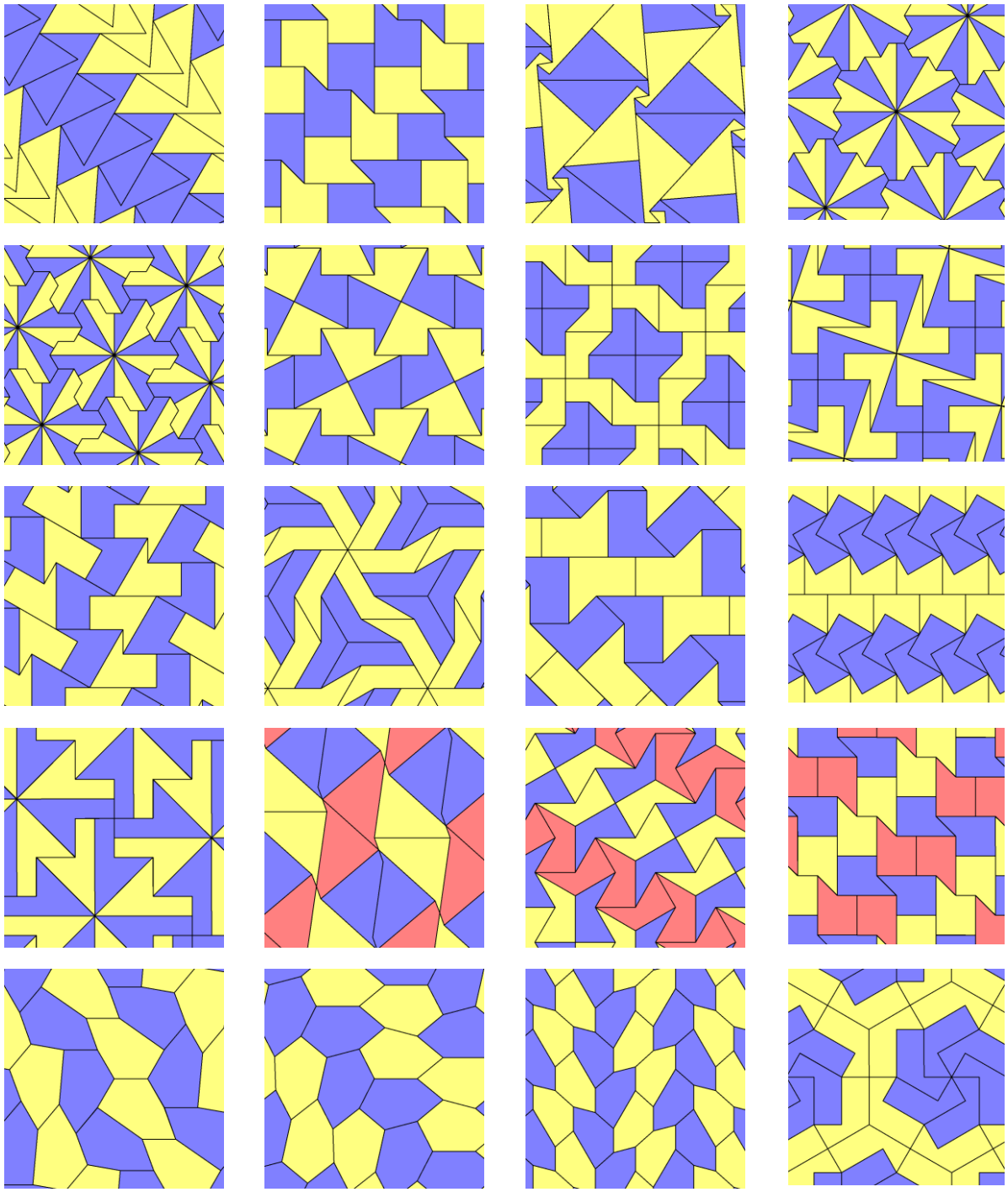


Type 8 special case

There are many fascinating tilings with triangles, quadrangles, or non-convex pentagons or even hexagons. To finish, here is a selection of the more interesting or beautiful ones. You can find all these tilings in a Java applet on my website. (4)







1. **Gardner, Martin.** *Time Travel and Other Mathematical Bewilderments.* sl : W. H. Freeman and Co., 1988. pp. 163-176.

2. **Grünbaum, B. and Shephard, G. C.** *Tilings and Patterns.* sl : W. H. Freeman and Co., 1987.

3. **Rice, Marjorie.** Intriguing Tesselations. *Intriguing Tesselations.* [Online] http://tessellations.home.comcast.net/~tessellations/.

4. **Scherphuis, Jaap.** *Jaap's Puzzle Page.* [Online] 2012. http://www.jaapsch.net/tilings.