Rotation Puzzles on Graphs
By Jaap Scherphuis

I collect permutation puzzles, in particular the Rubik's Cube and its many varied relatives. My collection also contains many sliding piece puzzles. In a paper from 1974, Richard M. Wilson [1] thoroughly analysed sliding piece puzzles by abstracting and generalising them to graphs. As far as I know, no similar analysis exists for rotational permutation puzzles, and here I hope to partly fill that gap.

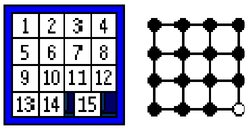

Figure 1

Figure 1 shows the Fifteen puzzle and its graph. The vertices represent the locations of the pieces, and two vertices are connected if a piece might slide between the two locations. To capture a rotational puzzle in a graph in a similar manner, we need some assumptions. First, the puzzle must have only one type of piece. The 3x3x3 Rubik's Cube for example has 2 types of pieces namely edges and corners, which don't intermingle, and a graph can represent only one of those. Furthermore, only their permutation will be examined, their orientations will be ignored. Lastly, a move consists of a turn of a face which cyclically permutes a set of pieces. Figure 2 shows how the edges of the Rubik's Cube are graphed.

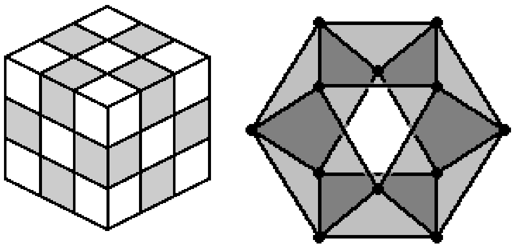

Figure 2

The 12 pieces of a Dino Cube [4], shown in figure 3, result in the same graph as that of the 12 Rubik's Cube edges. The difference is that the moves on a Dino Cube correspond to cycles on the triangular faces of the graph, whereas the moves on the Rubik's Cube correspond to cycles on the square faces. For a graph to represent a puzzle, certain faces must be marked in some way to show what moves are allowed on the puzzle.

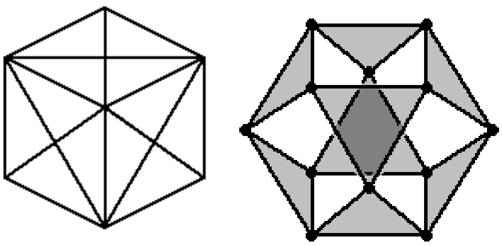

Figure 3

The question is now: Given a puzzle (as represented by a graph), which permutations of the pieces are possible?

Even and odd parities

In some puzzles every move is an even permutation. For example the Impossiball and the Alexander Star [4] have this property since every move is a 5-cycle. Obviously the best we can hope for in such cases is that all even permutations are possible. Suppose on the other hand that some move is an odd permutation, and furthermore, that we can already achieve every even permutation. Then all the odd permutations come for free:

Parity lemma: Let p be an odd permutation in S_n. Then $\langle A_n,p\rangle = S_n$.

The proof is almost trivial. Suppose you can solve any even permutation on a puzzle. If you are then given any odd permutation to solve, just do any odd permutation move. Such a move must exist for any odd permutation to be possible at all. This results in an even permutation which you can then solve.

From large to small

What remains now is to find a way to use the moves to produce all the even permutations. Puzzles with few possible moves tend to be harder because you have fewer degrees of freedom. Furthermore, if you can solve such a small puzzle, then it is easy to extend this knowledge to solve larger puzzles due to the following:

Extension lemma: Let An be the group of even permutations of n items (n>2). Let c be the cycle (k k+1 k+2 ... m), where k<=n<m. Then $\langle A_n,c\rangle$ contains Am.

Proof omitted. (Just establish the 3-cycles (1 2 i) for all i>2 in the three cases k>2, k=2, k=1).

In other words, if with just a few faces we can get every even permutation of their pieces, then by adding another adjacent face allows for every even permutation of this larger set of pieces. By adding the faces one by one, we can then get every even permutation of the whole puzzle. The way I have stated it here, it is assumed that only one section the added face overlaps with the previous faces, but with a little more work it can be shown that this assumption is not necessary - see Wilson's paper for details.

From this it is clear that we need to examine the smallest cases first, i.e. the graphs involving only two faces. If those two faces allow for all even permutations, then the extension lemma shows the whole puzzle allows all even permutations too.

Two-faced puzzles

At this point I will assume not only that the puzzle graph has only two faces, but also that the two faces have only one section in common. Any such graph puzzle can be specified using 3 numbers - the number of pieces unique to the first face (x), the number of pieces shared by the two faces (y), and the number of pieces unique to the second face (z). I will use the notation (x,y,z) to specify this graph.

The general (x,y,z) problem is split into many separate cases. Each time we establish a 3-cycle. Applying the extension lemma to the combination of that 3-cycle (or actually A_3) and a turn of an affected face, it follows that all even permutations in that face can be achieved. Another application of the lemma then shows that the two faces together allow every even permutation of the whole set of pieces.

Let's tackle some examples explicitly, for example the case x>0, y=1, z>0. The effect of the commutator $aba^{-1}b^{-1}$ on the pieces is already a 3-cycle (fig. 4). Note that I use left-to-right notation, i.e. permutation a is performed first, and b^{-1} last. Applying the extension lemma to this twice immediately shows that every even permutation is possible.

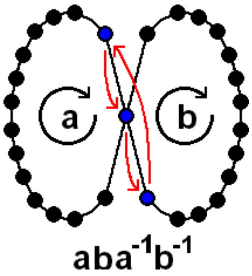

Figure 4

A more complicated case is $x>1$, $y>1$, $z>2$. The effect of the commutator $aba^{-1}b^{-1}$ on the pieces is now a double swap (fig. 5). The moves $a^{-1}b^{-2}$ would move two of the affected pieces out of the way in the unshared part of face b, and leave the other two affected pieces adjacent in face a. Therefore the conjugate $p = b^2a(aba^{-1}b^{-1})a^{-1}b^{-2}$ affects face a only by one swap of adjacent pieces. By commutating p with a, we get a 3-cycle of three adjacent pieces in face a. Thus every even permutation is possible.

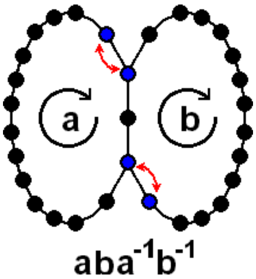

Figure 5

The various cases are listed in the table below, but the further details are omitted.

Case	**3-cycle**
(1+,1,1+)	$aba^{-1}b^{-1}$
(2+,2+,3+)	$p=b^2a(aba^{-1}b^{-1})a^{-1}b^{-2}$, $pap^{-1}a^{-1}$
(0,2+,1+)	$aba^{-1}b^{-1}$
(1,2+,1)	ab
(1,2,2+)	a
(1,3,3+)	$(b\,(aba^{-1}b^{-1})\,b^{-1}\,a^{-1}b^{-1}ab)^2$
(1,4+,2+)	$(ab^{-1}a^{-1}b\;\;a^{-1}bab^{-1})^2$
(2,3+,2)	$(aba^{-1}b^{-1})\;\;a^{-1}\,(a^{-1}b^{-1}ab)\,a\;\;(a^{-1}bab^{-1})^{-1}\,(a^{-1}b^{-1}ab)\,(a^{-1}bab^{-1})$

Cases (2,2,2) and (1,3,2)

All the previous graph puzzles allowed at least all even permutations to be performed on its pieces. The two special cases (2,2,2) and (1,3,2) however, shown in figure 6, only allow 5! positions instead of the expected 6! = 720. The (2,2,2) graph corresponds to a real puzzle, namely the puzzle formed by the corners of two adjacent face of a Rubik's Cube. David Singmaster [2] has given a proof that the group of permutations is $PGL_2(Z_5)$, a group of order 120 isomorphic to S_5.

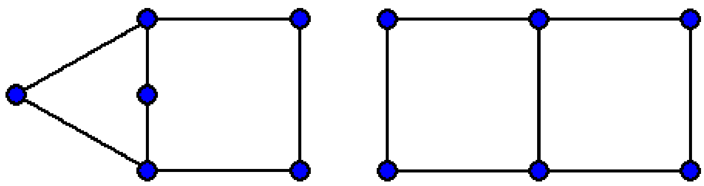

Figure 6

Suppose we add a third face to either of the special cases (2,2,2) and (1,3,2) so that we have 7 or more vertices in the graph. Are all even permutations possible then? If there is a subgraph with two faces that is not one of these two special cases, then we already have the answer, namely that they are possible. This leaves only a small number of possibilities to check, and it turns out that each time every even permutation can be achieved.

Comparison to sliding piece puzzles

Given an (x,y,z) graph, you can add a vertex in the shared section to turn it into an equivalent sliding piece puzzle. The added vertex is the initial location of the gap the other pieces can slide into. Turning one face of the original rotational puzzle corresponds to moving the space along the path around that face. It is therefore obvious that the positions of (x,y,z) rotational puzzle correspond exactly to the positions of the sliding puzzle with the gap at the inserted vertex.

The two exceptional rotational puzzles with graphs (2,2,2) and (1,3,2) can also be turned into sliding piece puzzles in this way. This results in the sliding puzzle graphs shown in figure 7a and 7b. In Richard M. Wilson's paper [1], only one exceptional graph was found (fig. 7d) and luckily his graph is isomorphic to both of these.

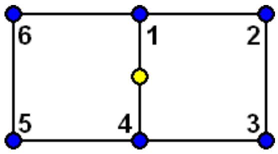

Figure 7a

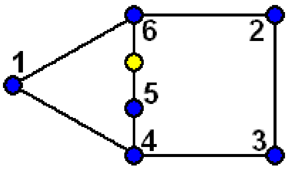

Figure 7b

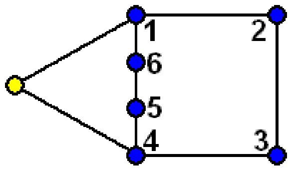

Figure 7c

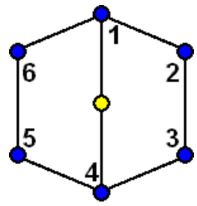

Figure 7d

Conclusion

Nearly every rotational puzzle graph allows every even permutation to be achieved. Furthermore, if any face contains an even number of pieces, then all odd permutations are also possible. Exceptions to this rule are:

- Trivially, the puzzle graphs with only one face (cyclic group)
- The two exceptional puzzle graphs ($PGL_2(Z_5) \sim S_5$)
- Puzzle graphs where any two adjacent faces always have more than one shared section. (group undetermined)

There are some interesting real-life examples in the third set of exceptions. A neat case is given by the three moves/faces (1 2 6 5), (1 3 6 4), and (2 3 5 4) which generate merely 24 positions (hint: think rolling die). The best known puzzle in this category is the Hungarian Rings [3].

References:

[1] Richard M. Wilson, **'Graph Puzzles, Homotopy, and the Alternating Group'**, J. Combin. Theory (Series B) 16 (1974) 86-96.

[2] David Singmaster, **Notes on Rubik's 'Magic Cube'**, Fifth edition, Enslow Publishers, 1980.

[3] David Singmaster, **'Cubic Circular'**, issue 5/6, 1982, p9, https://www.jaapsch.net/puzzles/cubic.htm.

[4] Jaap Scherphuis, **'Jaap's Puzzle Page'**, https://www.jaapsch.net/puzzles/